JOURNAL OF APPROXIMATION THEORY **14**, 115–127 (1975)

Characterization of Generalized Convex Functions by Best L^2-Approximations

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Let $\mathscr{M}$ be a normed linear space, and $\{M_n\}_1^\infty$ a sequence of increasing finite dimensional subspaces, i.e., $M_n \subset M_{n+1}$, for all n. For any element $f \in \mathscr{M}$, we obviously have

$$d(f, M_n) \geqslant d(f, M_{n+1}), \qquad \text{for all } n, \tag{1}$$

where $d(f, M_k)$ is the distance, in the metric induced by the norm, from M_k to f.

In a recent paper [2], we discussed the space $\mathscr{M} = C[a, b]$ with the uniform norm and with $M_n = [u_0, ..., u_{n-1}]$, the linear subspace spanned by $\{u_i\}_0^{n-1}$, where $\{u_i\}_0^\infty$ is an infinite Tchebycheff system. We established there that the functions for which inequality (1), for a given n, is strict for all subintervals of $[a, b]$ are precisely those that are convex with respect to $(u_0, u_1, ..., u_{n-1})$. The proof depended crucially on the alternance properties of the best approximants in the uniform norm. Somewhat surprisingly, analogous results are valid when the norm under consideration is the L^2-norm. In fact, as we show in this paper, generalized convex functions play the same role in the L^2-norm, for all continuous weights. Weaker results for the L^2 case were obtained in [6].

1. INTRODUCTION

Let $[a, b]$ be a fixed interval on the real line, n a positive integer. Denote by $V_n[a, b]$ the set of increasing vectors $(t_0, ..., t_n) \in [a, b]^{n+1}$, i.e., with $a \leqslant t_0 < t_1 < \cdots < t_n \leqslant b$, and by $V_n{}^*[a, b]$ the set of nondecreasing vectors $\in [a, b]^{n+1}$, i.e., with:

$$\begin{aligned} a \leqslant t_0 = \cdots = t_{\nu_0} < t_{\nu_0+1} = \cdots = t_{\nu_0+\nu_1+1} \\ < t_{\nu_0+\nu_1+2} = \cdots = t_{\nu_0+\cdots+\nu_k+k} \leqslant b, \end{aligned} \tag{1.1}$$

where the multiplicities $1 + \nu_i$ satisfy $\sum_{i=0}^{k} (1 + \nu_i) = n + 1$. For such $T \in V_n{}^*[a, b]$ we denote $\nu(T) = \max \nu_i$.

We define similarly $V_n(a, b)$ and $V_n{}^*(a, b)$ (with $a < t_0$, $t_n < b$). Let

$f_0, \ldots, f_n$ be real-valued functions on $[a, b]$, $T = (t_0, \ldots, t_n) \in V_n[a, b]$. Denote by

$$U\begin{pmatrix} f_0, \ldots, f_n \\ t_0, \ldots, t_n \end{pmatrix}$$

the determinant $|f_i(t_j)|_{i,j=0}^n$. $\mathscr{U} = \{u_0, \ldots, u_n\} \subset C[a, b]$ is called a *Tchebycheff system* over $[a, b]$ if

$$U\begin{pmatrix} u_0, \ldots, u_n \\ t_0, \ldots, t_n \end{pmatrix} > 0 \qquad \text{for all} \quad (t_0, \ldots, t_n) \in V_n[a, b].$$

Properties of such systems can be found in the comprehensive monograph [4]. We shall develop some of the properties which are useful for our purposes.

If $f_0, \ldots, f_n \in C^{(\nu)}[a, b]$ and $T = (t_0, \ldots, t_n) \in V_n{}^*[a, b]$ is with $\nu(T) \leqslant \nu$, we denote:

$$U^*\begin{pmatrix} f_0, \ldots, f_n \\ t_0, \ldots, t_n \end{pmatrix} = \begin{vmatrix} f_0(t_0) \cdots f_0^{(\nu_0)}(t_0) & f_0(t_{1+\nu_0}) \cdots f_0^{(\nu_n)}(t_{1+\nu_0}) \cdots f_0^{(\nu_k)}(t_n) \\ \vdots \qquad\qquad \vdots & \vdots \\ f_n(t_0) \cdots f_n^{(\nu_0)}(t_0) & \cdots f_n^{(\nu_k)}(t_n) \end{vmatrix}.$$

If

$$U^*\begin{pmatrix} u_0, \ldots, u_n \\ t_0, \ldots, t_n \end{pmatrix} > 0 \qquad \text{for all} \quad T = (t_0, \ldots, t_n) \in V_n{}^*[a, b],$$

then $\mathscr{U}$ is called an *extended Tchebycheff system* over $[a, b]$. An extended Tchebycheff system $\mathscr{U} = \{u_0, \ldots, u_n\}$ is called *complete* if $\{u_0, \ldots, u_j\}$ is an extended Tchebycheff system over $[a, b]$ for each $j = 0, 1, \ldots, n$.

It is shown in [4, p. 379] that every extended complete Tchebycheff system (ECT-system, for short) over $[a, b]$ satisfying the initial conditions: $u_k^{(j)}(a) = 0$ ($j = 0, 1, \ldots, k-1$; $k = 1, 2, \ldots, n$) is of the form:

$$\begin{aligned} u_0(t) &= w_0(t) \\ u_1(t) &= w_0(t) \int_a^t w_1(\xi_1)\, d\xi_1 \\ &\;\vdots \\ u_n(t) &= w_0(t) \int_a^t w_1(\xi_1) \int_a^{\xi_1} w_2(\xi_2) \int_a^{\xi_2} \cdots \int_a^{\xi_{n-1}} w_n(\xi_n)\, d\xi_n \cdots d\xi_1 \end{aligned} \tag{1.2}$$

where $w_i \in C^{(n-i)}[a, b]$ and $w_i > 0$ on $[a, b]$. With no loss of generality we may assume henceforth that $w_0 = 1$.

Note that the classical ECT-system on $[0, b]$: $1, t, t^2/2, \ldots, t^n/n!$ corresponds to $w_0(t) = \cdots = w_n(t) = 1$.

Given a fixed ECT-system $\mathscr{U}$ with the representation (1.2), we define the differential operators:

$$D_j f(t) = \frac{d}{dt}\left(\frac{f(t)}{w_j(t)}\right) \qquad (j = 0, \ldots, n),$$

$$L_{-1} f = f, \qquad L_j f = D_j D_{j-1} \cdots D_0 f, \qquad j = 0, 1, \ldots, n$$

and their adjoints $D_j^* f = (1/w_j(t))(df(t)/dt)$, $L_j^* f = D_0^* \cdots D_j^* f$, $j = 0, 1, \ldots, n$. Much of the theory of power series expansion can be easily generalized to ECT-systems of this form, e.g.:

PROPOSITION 1.1. (generalized Rolle's theorem): *If $f \in C[a, b] \cap C^{(1)}(a, b)$ and $f(a) = f(b) = 0$ then, for every $j = 0, 1, \ldots, n-1$, there is some $c_j \in (a, b)$ with $D_j f(c_j) = 0$.*

PROPOSITION 1.2. (generalized Taylor's theorem): *If $f \in C^{(n)}[a, b]$ then there is some $c \in (a, b)$ with*

$$f(b) = \sum_{j=0}^{n-1} \frac{L_{j-1} f(a)}{w_j(a)} u_j(b) + \frac{L_{n-1} f(c)}{w_n(c)} u_n(b).$$

(This is proved by applying the generalized Rolle theorem successively to

$$g(x) = f(x) - \sum_{j=0}^{n-1} \frac{L_{j-1} f(a)}{w_j(a)} u_j(x) - \frac{u_n(x)}{u_n(b)}\left[f(b) - \sum_{j=0}^{n-1} \frac{L_{j-1} f(a)}{w_j(a)} u_j(b)\right],$$

$L_0 g(x)$, etc.)

If f is a "u-polynomial" $f = \sum_{j=0}^{n-1} a_j u_j$, then this is its Taylor's expansion, i.e., $a_j = (L_{j-1} f(a))/(w_j(a))$.

For $f_0, \ldots, f_n \in C^{(\nu)}[a, b]$, $T = (t_0, \ldots, t_n) \in V_n^*[a, b]$ with $\nu(T) \leqslant \nu$, define:

$$U_u^* \begin{pmatrix} f_0, \ldots, f_n \\ t_0, \ldots, t_n \end{pmatrix} = \begin{vmatrix} f_n(t_0) \cdots L_{\nu_0 - 1} f_n(t_0) & f_n(t_{1+\nu_0}) \cdots L_{\nu_k - 1} f_n(t_n) \\ \vdots & \vdots \\ f_n(t_0) & \cdots \qquad L_{\nu_k - 1} f_n(t_n) \end{vmatrix}.$$

It is immediate to verify that

$$U_u^* \begin{pmatrix} u_0, \ldots, u_n \\ t_0, \ldots, t_n \end{pmatrix} = U^* \begin{pmatrix} u_0, \ldots, u_n \\ t_0, \ldots, t_n \end{pmatrix}.$$

For $T = (t_0, \ldots, t_n) \in V_n{}^*[a, b]$ we write $f(t_i) \overset{T}{=} 0$ $(i = 0, \ldots, n)$ if $f^{(j)}(t_r) = 0$, $r = 0, \ldots, k$; $j = 0, \ldots, \nu_r - 1$.

If we denote the linear span of $\mathscr{U}$ by $\Lambda(u_0, \ldots, u_n)$, then for every function $f \in C^{(\nu)}[a, b]$ and every $T = (t_0, \ldots, t_n) \in V_n{}^*[a, b]$ with $\nu(T) \leqslant \nu$, there is a unique "polynomial" $P = I_u(f, T) \in \Lambda(u_0, \ldots, u_n)$ interpolating f at T, i.e., satisfying $(P - f)(t_i) \overset{T}{=} 0$ $(i = 0, \ldots, n)$—this is $P(t) = \sum_{i=0}^{n} a_i u_i(t)$ where

$$a_i = U_u{}^* \begin{pmatrix} u_0, \ldots, u_{i-1}, f, u_{i+1}, \ldots, u_n \\ t_0 \qquad \ldots, \qquad t_n \end{pmatrix} \Big/ U_u{}^* \begin{pmatrix} u_0, \ldots, u_n \\ t_0, \ldots, t_n \end{pmatrix}.$$

We need the following two simple observations concerning these generalized interpolating polynomials:

PROPOSITION 1.3. *If $f \in C^{(\nu)}(a, b)$ has at t a zero of generalized order ν, i.e., if $L_j f(t) = 0$ for $j = -1, \ldots, \nu - 2$ but $L_{\nu-1} f(t) \neq 0$, then passing through t, f changes sign if ν is odd and preserves sign if ν is even.*

(Immediate from the generalized Taylor theorem.)

PROPOSITION 1.4. *If $f \in C^{(\nu)}(a, b)$, $T_j = (t_0{}^j, \ldots, t_n{}^j) \in V_n{}^*(a, b)$ $(j = 0, 1, 2, \ldots)$ with $\nu(T_j) \leqslant \nu$ and such that $T_j \to T_0$, then $I_u(f, T_j) \to I_u(f, T_0)$.*

Proof. It suffices to prove the case when the T_j differ only in one coordinate, and we can assume this coordinate to be the first one: $t_0{}^j \to t_0{}^0$, $t_i{}^j = t_i{}^0$ $(i = 1, \ldots, n)$. If $t_0{}^0 < t_1{}^0$, we may assume also $t_0{}^j < t_1{}^0$ so that

$$U_u{}^* \begin{pmatrix} u_0, \ldots, u_{i-1}, f, u_{i+1}, \ldots, u_n \\ t_0{}^j, t_1{}^0 \qquad \ldots, \qquad t_n{}^0 \end{pmatrix} \to U_u{}^* \begin{pmatrix} u_0, \ldots, u_{i-1}, f, u_{i+1}, \ldots, u_n \\ t_0{}^0, t_1{}^0 \qquad \ldots \qquad t_n{}^0 \end{pmatrix}$$

by the continuity of the u_k and f, and similarly

$$U^* \begin{pmatrix} u_0, \ldots, u_n \\ t_0{}^j, t_1{}^0, \ldots, t_n{}^0 \end{pmatrix} \to U^* \begin{pmatrix} u_0, \ldots, u_n \\ t_0{}^0, \ldots, t_n{}^0 \end{pmatrix}$$

(and this is nonzero since $\mathscr{U}$ is an ECT-system).

If $t_0{}^0 = t_1{}^0 = \cdots = t_k{}^0 < t_{k+1}^0$, we may assume that $t_0{}^0 \neq t_0{}^j < t_{k+1}^0$ for all $j \geqslant 1$, so that:

$$\frac{U_u{}^* \begin{pmatrix} u_0, \ldots, u_{i-1}, f, u_{i+1}, \ldots, u_n \\ t_0{}^j, t_1{}^0 \qquad \ldots, \qquad t_n{}^0 \end{pmatrix}}{U^* \begin{pmatrix} u_0, \ldots, u_n \\ t_0{}^j, t_1{}^0, \ldots, t_n{}^0 \end{pmatrix}} = \frac{\begin{vmatrix} u_0(t_0{}^j) & u_0(t_1{}^0) & \cdots & L_{k-2}u_0(t_1{}^0) & \cdots \\ & & \vdots & & \\ f(t_0{}^j) & f(t_1{}^0) & \cdots & L_{k-2}f(t_1{}^0) & \cdots \\ & & \vdots & & \\ u_n(t_0{}^j) & u_n(t_1{}^0) & \cdots & L_{k-2}u_n(t_1{}^0) & \cdots \end{vmatrix}}{\begin{vmatrix} u_0(t_0{}^j) & u_0(t_1{}^0) & \cdots & L_{k-2}u_0(t_1{}^0) & \cdots \\ & & \vdots & & \\ u_n(t_0{}^j) & u_n(t_1{}^0) & \cdots & L_{k-2}u_n(t_1{}^0) & \cdots \end{vmatrix}}$$

Applying the generalized Taylor theorem and the elementary operations on determinants, the elements $u_0(t_0{}^j),\ldots, u_n(t_0{}^j), f(t_0{}^j)$ in the first rows can be replaced, respectively, by

$$\frac{L_{k-1}u_0(c_0{}^j)}{w_k(c_0{}^j)}\, u_k(t_0{}^j),\ldots, \frac{L_{k-1}u_n(c_n{}^j)}{w_k(c_n{}^j)}\, u_k(t_0{}^j), \frac{L_{k-1}f(c^j)}{w_k(c^j)}\, u_k(t_0{}^j),$$

where

$$c_0{}^j, c_1{}^j,\ldots, c_n{}^j, c^j \in (t_0{}^j, t_1{}^0),$$

and by the continuity assumptions the quotient of the determinants tends to

$$\frac{\begin{vmatrix} \dfrac{L_{k-1}u_0(t_1{}^0)}{w_k(t_1{}^0)} & u_0(t_1{}^0) \cdots L_{k-2}u_0(t_1{}^0) \cdots \\ & \vdots \\ \dfrac{L_{k-1}f(t_1{}^0)}{w_k(t_1{}^0)} & f(t_1{}^0) \cdots L_{k-2}f(t_1{}^0) \cdots \\ & \vdots \\ \dfrac{L_{k-1}u_n(t_1{}^0)}{w_k(t_1{}^0)} & u_n(t_1{}^0) \cdots L_{k-2}u_n(t_1{}^0) \cdots \end{vmatrix}}{\begin{vmatrix} \dfrac{L_{k-1}u_0(t_1{}^0)}{w_k(t_1{}^0)} & u_0(t_1{}^0) \cdots L_{k-2}u_0(t_1{}^0) \cdots \\ & \vdots \\ \dfrac{L_{k-1}u_n(t_1{}^0)}{w_k(t_1{}^0)} & u_n(t_1{}^0) \cdots L_{k-2}u_0(t_1{}^0) \cdots \end{vmatrix}} = \frac{U_u{}^*\begin{pmatrix} u_0 ,\ldots, u_{i-1}, f, u_{i+1}, u_n \\ t_0{}^0, t_1{}^0,\ldots, t_n{}^0 \end{pmatrix}}{U^*\begin{pmatrix} u_0 ,\ldots, u_n \\ t_0{}^0,\ldots, t_n{}^0 \end{pmatrix}}$$

Q.E.D.

A function f defined on (a, b) is called *convex with respect to* $(u_0 ,\ldots, u_{n-1})$ if

$$U\begin{pmatrix} u_0 ,\ldots, u_{n-1}, f \\ t_0 ,\ldots, t_{n-1}, t_n \end{pmatrix} \geqslant 0,$$

for all $(t_0 ,\ldots, t_n) \in V_n(a, b)$. We denote the convex cone of functions convex with respect to $(u_0 ,\ldots, u_{n-1})$ by $C(u_0 ,\ldots, u_{n-1})$.

Notation. We denote by Λ_{n-1} the n-dimensional linear space spanned by $(u_0 ,\ldots, u_{n-1})$. We further denote by $T^2_{n-1}([\alpha, \beta]\ f)$; $[T^2_{n-1}([a, b]; f) = T^2_{n-1}(f)]$ the best approximant, in the L^2-norm with weight $w(x)$, where $w(x) > 0$ is a positive continuous function on $[a, b]$, on $[\alpha, \beta]$, from Λ_{n-1} to f. It is well known that there exists a unique best approximant. We let

$$E^2_{n-1}([\alpha, \beta]; f) = \| f - T^2_{n-1}([\alpha, \beta]; f)\|_2 ; \qquad E^2_{n-1}([a, b]; f) = E^2_{n-1}(f)$$

be the distance, in the L^2-norm, from Λ_{n-1} to f. Let $\{P_n[\alpha, \beta]\}_0^\infty$ (with $P_n[a, b] = P_n$) be the orthonormal system constructed from the u_i's by the

Gram-Schmidt process, normalized so that $P_n(b) > 0$. We define $a_n^2([\alpha, \beta]; f)$ $[a_n^2([a, b]; f) = a_n^2(f)]$ by

$$a_n^2([\alpha, \beta]; f) = \int_\alpha^\beta f(t)\, P_n[\alpha, \beta](t)\, w(t)\, dt \tag{1.3}$$

i.e., a_n^2 is the nth Fourier-Stieltjes coefficient.

2. Direct Theorems

We consider a fixed interval $[a, b]$, and establish properties of the best approximants in $L^2(w; [a, b])$ from Λ_{n-1} to functions of $C(u_0, ..., u_{n-1})$, where $w(x) > 0$ is a positive continuous weight function. It is well known that the best approximant, in the L^2-norm, from Λ_n to f, is given by

$$T_n^2(f) = \sum_{k=0}^{n} a_k^2 P_k(x), \tag{2.1}$$

where a_k^2 is the Fourier-Stieltjes coefficient of f with respect to P_k.

We now recall [7] that $P_n(x)$ induces a measure of the cone dual to $C(u_0, ..., u_{n-1})$, implying the following.

THEOREM 2.1. [7] *Let f be a function of $C(u_0, ..., u_{n-1})$. Then*

$$a_n^2 \geq 0. \tag{2.2}$$

If $f \in C(u_0, ..., u_{n-1}) \setminus \Lambda_{n-1}$ then

$$a_n^2 > 0. \tag{2.3}$$

The second part of the theorem does not appear explicitly in [7], but is easily deducible. Noting that

$$E_n^2(f) = E_{n-1}^2(f) \qquad \text{iff} \quad a_n^2 = 0. \tag{2.4}$$

we obtain Theorem 2.2.

THEOREM 2.2. *Let f be a function of $C(u_0, ..., u_{n-1}) \setminus \Lambda_{n-1}$. Then*

$$E_n^2(f) < E_{n-1}^2(f). \tag{2.5}$$

Another property which holds for generalized convex functions is expressed by Theorem 2.3.

THEOREM 2.3. *Let f be a function of* $C(u_0,..., u_{n-1})\backslash \Lambda_{n-1}$. *Then* $f - T^2_{n-1}(f)$ *has exactly n sign changes, and the last sign is* $(+)$.

Proof. Since $f - T^2_{n-1}(f)$ belongs to $C(u_0,..., u_{n-1})$, it can obviously have no more than n sign changes. Indeed, otherwise an appropriate choice of points $\{t_i\}_0^m$ would render the determinant

$$U\begin{pmatrix} u_0,..., u_{n-1}, f - T^2_{n-1}(f) \\ t_0,..., t_{n-1}, t_n \end{pmatrix} \tag{2.6}$$

negative.

On the other hand, in view of (2.1) and (1.3), we have

$$\int_a^b [f - T^2_{n-1}(f)]\, P_i w\, dx = 0, \qquad i = 0, 1,..., n-1.$$

Hence,

$$\int_a^b [f - T^2_{n-1}(f)]\, u_i w\, dx = 0, \qquad i = 0, 1,..., n-1. \tag{2.7}$$

Relation (2.7) implies, as in [4, p. 410], that $f - T^2_{n-1}(f)$ possesses at least n sign changes. Thus, $f - T^2_{n-1}(f)$ has exactly n sign changes. Returning now to the determinant (2.6), we conclude that the last sign is $(+)$. Q.E.D.

For generalized absolutely monotone functions (see [1 and 5]) we easily derive from Theorems 2.1 and 2.2 the following theorem.

THEOREM 2.4. *Let f be a generalized absolutely monotone function on* (a, b), *which does not coincide with a u-polynomial. Then*

1. $a_k^2 > 0$, *for all k*,
2. $\{E_n^2(f)\}_0^\infty$, *is a strictly decreasing sequence.*

3. CONVERSE THEOREMS

There exists no direct converse to the theorems of this section. This will be established (as in [2]) by general category arguments.

LEMMA 3.1. *Let* $r < s$ *be any real numbers, and n any positive integer. Let*

$$\mathscr{B}_n^2(r, s) = \{f;\, E^2_{n-1}([r, s]; f) = E_n^2([r, s]; f)\}.$$

Then

$$\mathscr{B}_n^2(r, s) \text{ is closed and has no interior.}$$

Proof. Let f_0 be an arbitrary function of $\mathscr{B}_n^2(r, s)$, and let ϵ be given.

Since $E^2_{n-1}([r, s]; f_0) = E_n^2([r, s]; f_0)$ and the best L^2-approximant is unique, we conclude that

$$q = T_n^2([r, s]; f_0) \in A_{n-1} .$$

Consider now the function

$$g = f_0 + \frac{\epsilon}{2} \frac{u_n}{\| u_n \|_\infty} .$$

Clearly, $\| g - f_0 \|_\infty < \epsilon$, and

$$T_n^2([r, s]; g) = q + \frac{\epsilon}{2} \frac{u_n}{\| u_n \|_\infty} \in A_n \setminus A_{n-1} .$$

Hence, $g \notin \mathscr{B}_n^2(r, s)$. Q.E.D.

THEOREM 3.2. *Let A be the set of functions such that $E_n^2([r, s]; f)$ is a strictly decreasing sequence for all n and all r, s rational in $[a, b]$. Then A^c is of the first category.*

Proof. We need only observe that

$$A^c = \bigcup_{n=1}^{\infty} \bigcup_{r,s \text{ rational}} \mathscr{B}_n^2(r, s)$$

and use the lemma.

A well known result (see [3], p. 260) implies that $D = \{f; f \in C[a, b]$, the right-hand derivative is finite for some $x \in [0, 1]\}$, is of the first category in $C[0, 1]$. Since we know (see [4], p. 385) that each $f \in C(u_0, ..., u_{n-1})$, $n \geq 2$ possesses a right hand derivative, it follows that $C(u_0, ..., u_{n-1})$, $n \geq 2$ is of the first category, hence much smaller than A.

We shall now prove that in spite of the foregoing analysis, properties of the type considered in Section 2 can be used to provide a characterization of generalized convexity cones. We note that if n has n continuous derivatives, the theorems are quite easy. However, the standard limit processes are of no avail, necessitating the following delicate argument, which involves coalescing points. We remark that the approach here is quite different from the one encountered in the uniform norm case [2], where the preservation of the separation of extremal points in the limiting process was decisive.

Let $\{v_0, ..., v_{2n-1}\}$ be the ECT-system on $[a, b]$ generated by $\{w_0, ..., w_{n-1}, ..., 1, w_{n-1}, ..., w_1\}$. (See [4] p. 528).

LEMMA 3.3. *Let g be a function with n continuous derivatives such that $g \notin C(v_0, ..., v_{2n-1})$. Then there exist a v-polynomial $Q \in A(v_0, ..., v_{2n-1})$ and a set of points $T \in V^*_{2n-1}(a, b)$ of the form*

$$t_0 = t_1 = \cdots = t_{n-1} < t_n = t_{n+1} = \cdots = t_{2n-1} , \tag{3.1}$$

such that

$$(g - Q)(t_i) \overset{T}{=} 0, \qquad i = 0, 1, \ldots, 2n-1,$$
$$(-1)^n(g - Q)(t) < 0, \qquad t_{n-1} < t < t_n . \tag{3.2}$$

Proof. Since $g \notin C(v_0, \ldots, v_{2n-1})$, there exist $(t_0, \ldots, t_{2n-1}) \in V_{2n-1}(a, b)$ and $\tau \in (t_{n-1}, t_n)$ such that

$$U\begin{pmatrix} v_0, \ldots, v_{2n-1}, g \\ t_0, \ldots, t_{n-1}, \tau, t_n, \ldots, t_{2n-1} \end{pmatrix} < 0. \tag{3.3}$$

Let $Q = T(g, T) \in \Lambda(v_0, \ldots, v_{2n-1})$. Since $(g - Q)(t_i) \overset{T}{=} 0, i = 0, 1, \ldots, 2n-1$ and

$$U\begin{pmatrix} v_0, \ldots, v_{2n-1}, Q \\ t_0, \ldots, t_n, \tau, t_n, \ldots, t_{2n-1} \end{pmatrix} = 0,$$

we deduce that $(-1)^n(g - Q)(\tau) < 0$.

Replace t_{n-1} and t_n, if necessary, by the nearest zeros (from the left and from the right, respectively) to τ, so that $(-1)^n(g - Q)(t) < 0$ in (t_{n-1}, t_n).

If the order, ν, of the zero of $g - Q$ at t_n is at least n, we replace $t_{n+1}, \ldots, t_{2n-1}$ by t_n and obtain

$$t_0 < t_1 < \cdots < t_n = t_{n+1} = \cdots = t_{2n-1},$$
$$(g - Q)(t_i) \overset{T}{=} 0 \qquad (i = 0, 1, \ldots, 2n-1), \tag{3.4}$$
$$(g - Q)(t) < 0 \qquad (t_{n-1} < t < t_n).$$

If $\nu < n$ we replace $t_{n+1}, \ldots, t_{n+\nu}$ by t_n. In this case t_n is an isolated zero of $g - Q$ (immediate from Rolle's theorem) and we take as $t_{n+\nu+1}$ the next zero of $g - Q$, again with its multiplicity, etc. We get thus what we call an "admissible" vector $\hat{T} = (\hat{t}_n, \hat{t}_{n+1}, \ldots, \hat{t}_{2n-1}) \in V^*_{n-1}(\tau, b)$, i.e., such that $\tilde{T} = (t_0, \ldots, t_{n-1}, \hat{t}_n, \hat{t}_{n+1}, \ldots, \hat{t}_{2n-1})$ and $Q = I_v(g, \tilde{T})$ satisfy $(-1)^n(g - Q) < 0$ on $(t_{n-1}, \hat{t}_n)$ and such that all zeros of $(g - Q)$ in $(\tau, \hat{t}_{2n-1})$ are properly counted (with their multiplicities).

We order the set of admissible vectors by: $\hat{T}_1 \leqslant \hat{T}_2$ if $\hat{T}_1$ precedes $\hat{T}_2$ in the lexicographic order and $\hat{t}^2_{2n-1} \leqslant \hat{t}^1_{2n-1}$. If $\hat{t}_n = \hat{t}_{n+1} = \cdots = \hat{t}_{2n-1}$, then $\hat{T}$ is obviously maximal with respect to this order. The fact that the converse also holds will follow immediately from the following observation: suppose $\hat{T} = (\hat{t}_n, \ldots, \hat{t}_{2n-1})$ is an admissible vector with $\hat{t}_{n+j} < \hat{t}_{n+j+1}$ for some $0 \leqslant j \leqslant n-2$, then there is an admissible $\hat{T}^1 = (\hat{t}_n^{\,1}, \ldots, \hat{t}^1_{2n-1})$ with $\hat{t}_i^{\,1} = \hat{t}_i$ for $i < n + j$, $\hat{t}_i^{\,1} \leqslant t_i$ for $i > n + j$, and $\hat{t}_{n+j} < \hat{t}^1_{n+j} < t_{n+j+1}$. In fact, take any $\eta \in (\hat{t}_{n+j}, \hat{t}_{n+j+1})$ and let Q^1 interpolate f at

$$T_\eta = (t_0, \ldots, t_{n-1}, \hat{t}_n, \ldots, \hat{t}_{n+j-1}, \eta, \hat{t}_{n+j+1}, \ldots, \hat{t}_{2n-1}),$$

i.e.,

$$Q^1 = Q + I_v(g - Q, T_n).$$

Since $\operatorname{sgn}(g - Q)(\eta) = (-1)^{n-j}$, $\operatorname{sgn} I_v(g - Q, T_n)$ is $(-1)^n$ on $(t_{n-1}, \hat{t}_n)$, so that for

$$t \in (t_{n-1}, \hat{t}_n): (-1)^n(g - Q^1)(t) = (-1)^n(g - Q)(t) - (-1)^n I_v(g - Q, T_n) < (-1)^n(g - Q)(t) < 0.$$

Let $\hat{t}_n^1, \dots, \hat{t}_{2n-1}^1$ be the proper counting of the zeros of $g - Q^1$ after t_{n-1}.

We want to show now the existence of a maximal admissible $\hat{T}$: Let $A = \{\hat{T}_\alpha; \alpha \in A\}$ be the set of all admissible $\hat{T}_\alpha = (t_n^\alpha, \dots, t_{2n-1}^\alpha)$ satisfying $\hat{T}_\alpha \geqslant \hat{T} = (t_n, \dots, t_{2n-1})$ (our first admissible vector). Let $t_n^0 = \sup_{\alpha \in A} t_n^\alpha$. If $t_n^0 = t_n^\alpha$ for some $\alpha \in A$ then by the above observation we must have $t_n^\alpha = t_{n+1}^\alpha = \cdots = t_{2n-1}^\alpha$ (otherwise, by repeated application of the procedure, if necessary, we get an admissible $\hat{T}_\beta$ $(\beta \in A)$ with $t_n^\beta > t_n^\alpha$. If $t_n^0 > t_n^\alpha$ for all $\alpha \in A$, take a sequence $\hat{T}_{\alpha_m}$ with $t_n^{\alpha_m}$ increasing to t_n^0. By passing to subsequences, if necessary, we may assume also that the limits $t_j^0 = \lim_{m\to\infty} t_j^{\alpha_m}$ exist for $j = n + 1, \dots, 2n - 1$, i.e., $\hat{T}_{\alpha_m} \to \hat{T}_0 = (t_n^0, \dots, t_{2n-1}^0)$. By Proposition 1.4 $I_u(g, T_{\alpha_m}) \to I_u(g, T_0)$. By the observation above $|I_v(g, T_{\alpha_m})|$ is increasing on $(t_{n-1}, t_n^{\alpha_k})$ for $m > k$, so that $t_n^0 = t_n^\alpha$ for some $\alpha \in A$. A similar treatment of the left end-points completes the proof of the lemma.

LEMMA 3.4. *Let $f \in C[a, b]$ be such that $f \notin C(u_0, \dots, u_{n-1})$. Then there exists a subinterval $[\alpha, \beta] \subset [a, b]$ such that*

$$\int_\alpha^\beta f(t)\, \tilde{P}_n[\alpha, \beta](t)\, dt < 0 \tag{3.5}$$

where $\tilde{P}_n[\alpha, \beta]$ is the n-th orthonormal u-polynomial on $[\alpha, \beta]$ with respect to the weight function 1, *and with positive highest coefficient which we denote by b_n.*

Proof. Let $(v_0, \dots, v_{2n-1})$ be the ECT-system generated by $\{1, w_{n-1}, \dots, w_1, 1, w_1, \dots, w_{n-1}\}$, i.e., corresponding to the differential operator $D_{n-1} \cdots D_1 D_0 D_0^* \cdots D_{n-1}^*$. Let g be any solution of $D_0^* \cdots D_{n-1}^* g = f$. Since $f \notin C(u_0, \dots, u_{n-1})$ it follows that $g \notin C(v_0, \dots, v_{2n-1})$. By Lemma 3.3 there exists a set $T \in V_{2n-1}^*(a, b)$ with $t_0 = t_1 = \cdots = t_{n-1} = \alpha$, $t_n = t_{n+1} = \cdots = t_{2n-1} = \beta$ such that $Q = I_v(g, T)$ satisfies

$$(-1)^n(g - Q)(t) < 0, \qquad \text{for} \quad t \in (\alpha, \beta). \tag{3.6}$$

Let $R = D_0^* \cdots D_{n-1}^* Q$. Since $L_{n-1}R = 0$, it follows that R is a function of A_{n-1}. Since $\tilde{P}_n[\alpha, \beta]$ is orthogonal to A_{n-1}, we have

$$\int_\alpha^\beta D_0^* \cdots D_{n-1}^* Q \tilde{P}_n[\alpha, \beta]\, dt = \int_\alpha^\beta R \tilde{P}_n[\alpha, \beta]\, dt = 0.$$

Hence, integrating repeatedly by parts, we have

$$\begin{aligned}\int_\alpha^\beta f\tilde{P}_n[\alpha,\beta]\,dt &= \int_\alpha^\beta D_0{}^* \cdots D_{n-1}^* g\tilde{P}_n[\alpha,\beta]\,dt \\ &= \int_\alpha^\beta D_0{}^* \cdots D_{n-1}^*(g-Q)\,\tilde{P}_n[\alpha,\beta]\,dt \\ &= D_1{}^* \cdots D_{n-1}^*(g-Q)\,\frac{\tilde{P}_n[\alpha,\beta]}{w_0}\Big|_\alpha^\beta \\ &\quad - \int_\alpha^\beta D_0\tilde{P}_n[\alpha,\beta]\,D_1{}^* \cdots D_{n-1}^*(g-Q)\,dt \\ &= -D_2{}^* \cdots D_{n-1}^*(g-Q)\,\frac{D_0\tilde{P}_n[\alpha,\beta]}{w_1}\Big|_\alpha^\beta \\ &\quad + \int_\alpha^\beta D_1D_0\tilde{P}_n[\alpha,\beta]\,D_2{}^* \cdots D_{n-1}^*(g-Q)\,dt \\ &= \cdots = b_n\int_\alpha^\beta (-1)^n(g-Q)\,dt.\end{aligned}$$

The integrated terms vanish by (3.2). Relation (3.6) therefore implies that (3.5) holds.

LEMMA 3.5. *Let $f \in C[a,b]$ be such that $f \notin C(u_0,\ldots,u_{n-1})$, and let $w(t) > 0$ be a continuous function. Then there exists a subinterval $[\alpha,\beta] < [a,b]$ such that*

$$\int_\alpha^\beta f(t)\,P_n[\alpha,\beta]\,w(t)\,dt < 0.$$

Proof. We introduce the new variable y defined by

$$\int_a^t w(s)\,ds = y(t) = y$$

and note that y is a strictly increasing, continuously differentiable function of t, so that its inverse exists and possesses similar properties.

Let now $f[t(y)] = h(y)$, $u_i[t(y)] = z_i(y)_0$ $P_n[\alpha,\beta](t) = R_n[\alpha,\beta](y)$. It is easy to see that $h \notin C(z_0,\ldots,z_{n-1})$, and that $R_n[\alpha,\beta](y)$ is equal to the nth orthonormal z-polynomial (with respect to the weight function 1) on the interval $[y(\alpha), y(\beta)]$.

Since $h \notin C(z_0,\ldots,z_{n-1})$ there exists, by Lemma 3.4 an interval $[A,B]$ in $[y(a), y(b)]$ such that,

$$\int_A^B h(y)\,R_n[y^{-1}(A), y^{-1}(B)](y)\,dy < 0.$$

Making the inverse change of variables, we obtain

$$\int_{h^{-1}(A)}^{h^{-1}(B)} f(t)\, P_n[y^{-1}(A), y^{-1}(B)](t)\, w(t)\, dt < 0.$$

Q.E.D.

THEOREM 3.6. *Let $f \in C[a, b]$. If $a_n^2([\alpha, \beta]; f) > 0$ for all $[\alpha, \beta] \subset [a, b]$, then $f \in C(u_0, ..., u_{n-1}) \backslash A_{n-1}$ for all $[\alpha, \beta]$.*

Proof. Clearly, f cannot coincide with an element of A_{n-1} on any interval, since this would imply $a_n^2([\alpha, \beta]; f) = 0$ for the interval under consideration. Furthermore, if $f \notin C(u_0, ..., u_{n-1})$, then by Lemma 3.5 there must exist an interval for which $a_n([\alpha, \beta]; f) < 0$, again violating the hypothesis in our Theorem. Q.E.D.

THEOREM 3.7. *Let $f \in C[a, b]$. If*

$$E_{n-1}^2([\alpha, \beta]; f) > E_n^2([\alpha, \beta]; f), \tag{3.7}$$

for all $[\alpha, \beta] \subset [a, b]$, then either f or $-f$ belong to $C(u_0, ..., u_{n-1}) \backslash A_{n-1}$ for all $[\alpha, \beta]$.

Proof. Since (3.7) is equivalent to $a_n^2([\alpha, \beta]; f) \neq 0$, f cannot belong to A_{n-1}. Since $a_n^2([\alpha, \beta]; f)$ is a continuous function of α and β and does not vanish, it must be of constant sign. The theorem follows now by an appeal to Theorem 3.6.

THEOREM 3.8. *Let $f \in C[a, b]$. If, for all $[\alpha, \beta] \subset [a, b]$, $f - T_{n-1}^2([\alpha, \beta]; f)$ has exactly n sign changes on $[\alpha, \beta]$ such that the last sign is $(+)$, then $f \in C(u_0, ..., u_{n-1}) \backslash A_{n-1}$ for all $[\alpha, \beta]$.*

Proof. Clearly f cannot coincide with a function of A_{n-1} on any interval. Assume now that $f \notin C(u_0, ..., u_{n-1})$. Then there exists, by Lemma 3.5, an interval $[\alpha_0, \beta_0]$ such that

$$\int_{\alpha_0}^{\beta_0} f P_n[\alpha_0, \beta_0]\, w\, dx < 0.$$

In view of the orthogonality conditions, we thus have

$$\int_{\alpha_0}^{\beta_0} [f - T_{n-1}^2([\alpha_0, \beta_0]; f)]\, P_n([\alpha_0, \beta_0])\, w\, dx < 0. \tag{3.8}$$

However, the pattern of sign changes taken together with the "moment conditions" (2.7) imply that $f - T_{n-1}^2([\alpha, \beta]; f)$ belongs to the dual cone $C^*(u_0, ..., u_{n-1})$ (see [4], p. 409) on $[\alpha_0, \beta_0]$. Since $P_n[\alpha_0, \beta_0]$ evidently belongs to $C(u_0, ..., u_{n-1})$, (3.8) is impossible.

CONCLUSION

Let f belong to $C[a, b]$. Then the following statements are equivalent:

(a) $E_{n-1}^2([\alpha, \beta]; f) > E_n^2([\alpha, \beta]; f)$, for all $[\alpha, \beta]$, $a \leqslant \alpha < \beta \leqslant b$.

(b) Either f or $-f$ belongs to $C(u_0, ..., u_{n-1}) \backslash \Lambda_{n-1}$ for all such $[\alpha, \beta]$.

(c) Either $a_n^2([\alpha, \beta]; f) > 0$ for all such $[\alpha, \beta]$, or $a_n^2([\alpha, \beta]; f) < 0$ for all such $[\alpha, \beta]$.

(d) $f - T_{n-1}^2([\alpha, \beta]; f)$ has exactly n sign changes on $[\alpha, \beta]$ for all such $[\alpha, \beta]$.

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